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Asymptotics of supremum distribution of $\alpha(t)$-locally stationary Gaussian processes

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Abstract

We study the exact asymptotics of $\mathbb{P}(\sup_{t\in[0,S]} X(t) > u)$, as $u \to \infty$, for centered Gaussian processes with the covariance function satisfying

$$1 - \mathbb{C}\text{ov}(X(t), X(t+h)) = A(t)|h|^{\alpha(t)} + o(|h|^{\alpha(t)}),$$

as $h \to 0$.

The obtained results complement those already considered in the literature for the case of locally stationary Gaussian processes in the sense of Berman, where $\alpha(t) \equiv \alpha$. It appears that the behavior of $\alpha(t)$ in the neighborhood of its global minimum on $[0, S]$ significantly influences the asymptotics.

As an illustration we work out the case of $X(t)$ being a standardized multifractional Brownian motion.

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1. Introduction

The analysis of the distribution of suprema of Gaussian processes, being a natural subject of interest in the extreme value theory, is continuously motivated by models that arise in applied probability; see [5,6,10,11] and references therein. The vast literature that deals with

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the distribution properties of extremes of Gaussian processes focuses on the analysis of

$$\mathbb{P}\left(\sup_{t\in[0,S]} X(t) > u\right), \tag{1}$$

where $\{X(t) : t \in [0, S]\}$ is a centered Gaussian process with a constant variance function, say equal to 1; see e.g. [3,4,8,13]. One of the reasons for the interest in this subclass of Gaussian processes is that the knowledge of properties of (1) allows us to study more complex problems. In particular, *the double sum* method enables application of asymptotic results for the supremum distribution of centered stationary Gaussian processes to study the supremum distribution of noncentered Gaussian processes with nonconstant variance function; see [5,10,11] or the proof of Theorem D.3 in [14].

In this context the class of *locally stationary* Gaussian processes with index α (introduced by Berman [3]) plays an important role. Recall that a centered Gaussian process $\{X(t) : t \in [0, S]\}$ with a constant variance function, say equal to 1, is called locally stationary with index α, if its covariance function satisfies

$$1 - \mathbb{C}\mathrm{ov}(X(t), X(t+h)) = \frac{1}{2}\mathbb{V}\mathrm{ar}(X(t) - X(t+h)) = A(t)|h|^{\alpha} + o(|h|^{\alpha}), \quad \text{as } h \to 0, \tag{2}$$

uniformly with respect to $t \in [0, S]$, where $\alpha \in (0, 2]$ and $A(t)$ is a bounded, strictly positive and continuous function. An important property of (2) is that it allows minor fluctuations of dependence at the global scale (by $A(t)$) and at the same time keeps the stationary structure at the local scale. We refer to [3,4,9] for studies on the asymptotics of (1) for locally stationary Gaussian processes with index α. Although the class of locally stationary processes with index α provides a rich source of Gaussian processes, it does not include all the processes that are currently of interest. In particular, one of the drawbacks is the assumption that index α is constant. This requirement excludes, for example, processes related to *multifractional Brownian motions*; [2,7,12].

This was a motivation for us to obtain the following extension of the class of locally stationary Gaussian processes with index α. We write $f(t) \in C(\mathcal{T})$ in order to denote that $f(t)$ is continuous on $\mathcal{T}$.

Definition 1.1. A real valued separable Gaussian process $\{X(t) : t \in [0, S]\}$ is said to be $\alpha(t)$-locally stationary if

(i) $\mathbb{E}X(t) = 0$, $\mathbb{V}\mathrm{ar}(X(t)) = 1$ for all $t \in [0, S]$;
(ii) $\alpha(t) \in C([0, S])$ and $\alpha(t) \in (0, 2]$ for all $t \in [0, S]$;
(iii) $A(t) \in C([0, S])$ and $0 < \inf\{A(t) : t \in [0, S]\} \le \sup\{A(t) : t \in [0, S]\} < \infty$;
(iv)

$$1 - \mathbb{C}\mathrm{ov}(X(t), X(t+h)) = A(t)|h|^{\alpha(t)} + o(|h|^{\alpha(t)}), \quad \text{as } h \to 0,$$

uniformly with respect to $t \in [0, S]$.

In this paper we focus on the exact asymptotics of supremum distribution of $\alpha(t)$-locally stationary Gaussian processes on $[0, S]$. It appears that the case of $\alpha(t)$ attaining its global minimum at a single point $t_o \in [0, S]$ is particularly interesting. Under this assumption the local behavior of $\alpha(t)$ in the neighborhood of t_0 influences the asymptotics. This scenario is analyzed in Theorem 2.1, which is the main result of the paper.

The idea of the proof of Theorem 2.1 is based on a modification of *double sum* technique. This method was developed by Pickands [13] for stationary Gaussian processes and then extended to other classes of Gaussian processes. We refer to [14] for the description and extensions of the double sum technique; see also [6]. While in the classical setting it is used to divide the analyzed interval on subintervals of comparable length, it appears that in the considered case it is crucial to accommodate lengths of the subintervals to the local behavior of $\alpha(t)$ in the neighborhood of t_0.

The reminder of the paper is organized as follows. The main result of the paper is presented in Section 2. In Section 3 we apply Theorem 2.1 to the class of standardized multifractional Brownian motions. The proof of the main result is given in Section 4.

In the following we use the following notation. For a given $a \in (0, 2]$ by $\mathcal{H}_a$ we denote the classical Pickands' constant, defined by the limit $\mathcal{H}_a = \lim_{T\to\infty} \frac{\mathcal{H}_a[T]}{T}$, where

$$\mathcal{H}_a[T] = \mathbb{E}\exp\left(\max_{t\in[0,T]}\left(\sqrt{2}B_{a/2}(t) - t^a\right)\right),$$

with $B_{a/2}(t)$ being a fractional Brownian motion with Hurst parameter $a/2$. Throughout the paper $\Psi(\cdot)$ denotes the complementary distribution function of the standard normal random variable. Recall that $\Psi(u) \sim \frac{1}{\sqrt{2\pi}u}e^{-u^2/2}$, as $u \to \infty$.

2. Main result

In this section we study the exact asymptotics of (1) for $\{X(t) : t \in [0, S]\}$ being an $\alpha(t)$-locally stationary Gaussian process. We distinguish between the following scenarios.

$\diamond$ *The function* $\alpha(t)$ *is constant on* $[a, b] \subset [0, S]$ *and attains its global minimum in* $[a, b]$.

In this case the problem can be reduced to the analysis of locally stationary process with index $\alpha = \alpha(a)$. Indeed, an application of Slepian's inequality (see, e.g. Theorem C.1 in [14]) yields

$$\mathbb{P}\left(\sup_{t\in[0,S]} X(t) > u\right) = \mathbb{P}\left(\sup_{t\in[a,b]} X(t) > u\right)(1 + o(1)),$$

as $u \to \infty$. Thus we can apply [4] to get the desired asymptotics.

$\diamond$ *The function* $\alpha(t)$ *attains its global minimum at a single point in* $[0, S]$.

In this case the analysis is nontrivial and the asymptotics significantly depends on the local behavior of $\alpha(t)$ in the neighborhood of the point minimizing the value of $\alpha(t)$. We make the following assumptions on function $\alpha(t)$:

A1. $\alpha(t) \in C([0, S])$ and attains its global minimum at the unique point $t_0 \in [0, S]$;

A2. there exist $B, \beta, \delta > 0$ such that

$$\alpha(t + t_0) = \alpha(t_0) + Bt^\beta + o(t^{\beta+\delta}), \quad \text{as } t \to 0.$$

In the following theorem we give the exact asymptotics of (1) under A1 and A2. For the sake of simplicity we focus on the case when $t_0 = 0$ or $t_0 = S$. The complementary scenario is when $t_0 \in (0, S)$ follows an analogous argumentation, giving the asymptotics twice as large as in the case considered here.

Theorem 2.1. *Let $\{X(t) : t \in [0, S]\}$ be an $\alpha(t)$-locally stationary Gaussian process that satisfies* A1 *and* A2. *If $t_0 = 0$ or $t_0 = S$, then*

$$\mathbb{P}\left(\sup_{t\in[0,S]} X(t) > u\right) = \mathcal{H}_\alpha A^{1/\alpha} \left(\frac{\alpha^2}{2B}\right)^{1/\beta} \frac{\Gamma(1/\beta)}{\beta} \frac{u^{2/\alpha}}{\log^{1/\beta}(u)} \Psi(u)(1 + o(1)),$$

as $u \to \infty$, where $\alpha = \alpha(t_0)$ and $A = A(t_0)$.

The proof of Theorem 2.1 is presented in Section 4.

Remark 2.1. The main part of the proof of Theorem 2.1 deals with the analysis of supremum distribution of $X(t)$ on an interval including t_0 and of length $t_u = \left(\frac{\alpha(t_0)^2}{\beta} \frac{\log(\log(u))}{\log(u)}\right)^{1/\beta}$. We note that although $t_u \to 0$ as $u \to \infty$ (and hence $\alpha(t) \to \alpha(t_0)$), the reduction of the problem to the locally stationary case with index $\alpha(t_0)$ gives incorrect asymptotics. The reason for this is that t_u is sufficiently “large” for the asymptotics to be influenced by the local behavior of $\alpha(t)$ in the neighborhood of t_0 (assumption A2).

This also influences the proof. Although the frame of the argumentation follows the procedure common for the double sum technique, the details differ from that of the standard cases. In the classical case (for instance if $X(t)$ is stationary) the asymptotics is obtained by a summation of the asymptotics on subintervals of comparable length that form a division of the considered interval. For the analyzed class of processes it is useful to make a 2-fold division of the analyzed interval. In the first step we find the division on subintervals of the length that is accommodated to parameter β in A2 (intervals A_k introduced in the proof). Then each interval A_k is divided on subintervals $B_{j,k}$ of the length related to the local value of $\alpha(t)$, $t \in A_k$. Finally, the asymptotics of supremum distribution on each $B_{j,k}$ is calculated (in this part an important role is played by Lemma 7 in [11], recently proved by Hüsler & Piterbarg).

3. Example

A rich source of $\alpha(t)$-locally stationary Gaussian processes is closely related with the class of *multifractional Brownian motions* (mfBm). Recall (see, e.g., [1]) that a multifractional Brownian motion $\{B_{H(t)}(t), t \geq 0\}$ is a centered Gaussian process with covariance function

$$\mathbb{E}[B_{H(t)}(t)B_{H(s)}(s)] = \frac{1}{2}D(H(s) + H(t))\Big[|s|^{H(s)+H(t)} + |t|^{H(s)+H(t)} - |t - s|^{H(s)+H(t)}\Big],$$

where $D(x) = \frac{2\pi}{\Gamma(x+1)\sin(\frac{\pi x}{2})}$ and $H(t)$ is a Hölder function of exponent γ such that $0 < H(t) < \min(1, \gamma)$ for $t \in [0, \infty)$. One of the main features of mfBms is that the regularity of sample paths is allowed to vary with t. In particular, for each t the trajectories are Hölder regular with exponent $H(t)$. We refer to [1,2,7,12] and references therein for the sake of properties and applications of mfBms in such areas as signal or image processing.

Define

$$\overline{B}_{H(t)}(t) := \frac{B_{H(t)}(t)}{\sqrt{\mathbb{V}\mathrm{ar} B_{H(t)}(t)}} \quad \text{for } t > 0.$$

Pretty lengthy, but standard, calculations show that for each $0 < \underline{S} < \overline{S} < \infty$ process $\{\overline{B}_{H(t)}(t) : t \in [\underline{S}, \overline{S}]\}$ is $2H(t)$-locally stationary with function $A(t) = \frac{1}{2}t^{-2H(t)}$ (see also [12] for other properties of $\overline{B}_{H(t)}(t)$).

Remark 3.1. The restriction that $0 < \underline{S}$ and $\overline{S} < \infty$ ensures that $A(t) = \frac{1}{2}t^{-2H(t)}$ satisfies (iii) of Definition 1.1 (see also examples of locally stationary processes in the sense of Berman given in [4]).

In the following corollary we straightforwardly apply Theorem 2.1 to standardized multifractional Brownian motions.

Corollary 3.1. *Let* $0 < \underline{S} < \overline{S} < \infty$ *and assume that:*

(1) $H(t)$ *is a Hölder function with exponent* γ*, such that* $0 < H(t) < \min(1, \gamma)$ *for any* $t \in [\underline{S}, \overline{S}]$*;*
(2) $H(t)$ *attains its minimum at the unique point* $t_0 \in \{\underline{S}, \overline{S}\}$*;*
(3) *there exist* $B, \beta, \delta > 0$ *such that* $H(t + t_0) = H(t_0) + Bt^\beta + o(t^{\beta+\delta})$*, as* $t \to 0$*. Then*

$$\mathbb{P}\left(\sup_{t\in[\underline{S},\overline{S}]} \overline{B}_{H(t)}(t) > u\right) = 2^{-1/2H}\frac{\mathcal{H}_{2H}}{t_0}\left(\frac{2H^2}{B}\right)^{1/\beta}\frac{\Gamma(1/\beta)}{\beta}\frac{u^{1/H}}{\log^{1/\beta}(u)} \times \Psi(u)(1+o(1)),$$

as $u \to \infty$*, with* $H = H(t_0)$*.*

Remark 3.2. Note that if we replace in (2) of Corollary 3.1 $t_0 \in \{\underline{S}, \overline{S}\}$ by $t_0 \in (\underline{S}, \overline{S})$, then the asymptotics is twice as large as for that presented in the Corollary 3.1 scenario.

Assumptions (1)–(3) of Corollary 3.1 imply that $\gamma \geq \beta$.

4. Proofs

Throughout this section we tacitly assume that $X(t)$ is an $\alpha(t)$-locally stationary Gaussian process such that $\alpha(t)$ satisfies A1 and A2. In the rest of the paper, without loss of generality, we suppose that $t_0 = 0$. Set $\alpha := \alpha(0)$ and $A := A(0)$.

The proof of Theorem 2.1 consists of two steps. In step 1 we find the asymptotics of $\mathbb{P}\left(\sup_{t\in[0,t_u]} X(t) > u\right)$, as $u \to \infty$, where

$$t_u = \left(\frac{\alpha^2}{\beta}\frac{\log(\log(u))}{\log(u)}\right)^{1/\beta}. \tag{3}$$

In step 2 (Lemma 4.6) we prove that, as $u \to \infty$,

$$\mathbb{P}\left(\sup_{t\in[t_u,S]} X(t) > u\right) = o\left(\mathbb{P}\left(\sup_{t\in[0,t_u]} X(t) > u\right)\right).$$

Before giving the proof of Theorem 2.1, we make some remarks that simplify the presentation:

- ◇ Without loss of generality we may suppose that A2 is satisfied with $B = 1$. Indeed, process $X_1(t) := X(B^{-1/\beta}t)$ is $\alpha_1(t)$-locally stationary with $\alpha_1(t) = \alpha(B^{-1/\beta}t)$ and $A_1(t) = B^{-\alpha(B^{-1/\beta}t)/\beta}A(B^{-1/\beta}t)$. Thus $\alpha_1(t)$ satisfies A1 and A2 in such a way that $\alpha_1(t) = \alpha + t^\beta + o(t^{\beta+\delta})$ as $t \to 0$.

◇ Since $t_u \to 0$ as $u \to \infty$ and $A(t)$ is continuous, then in the argumentation for step 1, without loss of generality, we may assume that $A(t) \equiv A(0) = A$. Moreover, function $A(t)$ does not play a significant role in the proof of step 2.

Hence, in the rest of the paper we assume that $A(t) \equiv A$ and $B = 1$, having in mind that for the asymptotics of the original process we have to make the substitution

$$A := B^{-\alpha/\beta} A(0). \tag{4}$$

Before proving Theorem 2.1, we need to introduce some notation and auxiliary results.

Let $T > 1$ be given. We introduce $a_k = a_k(u) := \left(\frac{k}{\log(u)(\log\log(u))^{1/\beta}}\right)^{1/\beta}$ and $A_k = A_k(u) := \left[a_k, a_{k+1}\right]$. Moreover let $m(u) := \lfloor \frac{\alpha^2}{\beta} (\log\log(u))^{1+1/\beta} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. Clearly we have

$$\sum_{k=0}^{m(u)-1} A_k \subset [0, t_u] \subset \sum_{k=0}^{m(u)} A_k.$$

We divide each interval A_k into subintervals of length $T/u^{2/\alpha(a_{k+1})}$. That is let

$$B_{j,k} = B_{j,k}(u) := \left[a_k + j \frac{T}{u^{2/\alpha(a_{k+1})}}, a_k + (j+1) \frac{T}{u^{2/\alpha(a_{k+1})}} \right]$$

for $j = 0, 1, \ldots, n(k)$, where $n(k) = n(k, u) := \lfloor \frac{a_{k+1} - a_k}{T} u^{2/\alpha(a_{k+1})} \rfloor$. Notice that

$$\sum_{j=0}^{n(k)-1} B_{j,k} \subset A_k \subset \sum_{j=0}^{n(k)} B_{j,k}.$$

Let

$$\mathcal{L} = \{(j, k) : j, k \in \mathbb{N}, 0 \le k \le m(u) - 1, 0 \le j \le n(k) - 1\}$$

and

$$\mathcal{U} = \{(j, k) : j, k \in \mathbb{N}, 0 \le k \le m(u), 0 \le j \le n(k)\}$$

so we have

$$\sum_{(j,k) \in \mathcal{L}} B_{j,k} \subset [0, t_u] \subset \sum_{(j,k) \in \mathcal{U}} B_{j,k}.$$

The following notation is useful in counting the ‘distance’ between segments of type $B_{j,k}$. For $(j_1, k_1), (j_2, k_2) \in \mathcal{L}$ we write

$$(j_1, k_1) \prec (j_2, k_2) \quad \text{iff} \quad (k_1 < k_2) \vee (k_1 = k_2 \wedge j_1 < j_2)$$

and define

$$N_{j_1,k_1}^{j_2,k_2} := \sharp\{(j, k) \in \mathcal{L} : (j_1, k_1) \prec (j, k) \prec (j_2, k_2)\}.$$

We use the following families of Gaussian processes to find tight bounds for supremum distribution of $X(t)$ on segments $B_{j,k}$. Let

- $\{Y_{\varepsilon,u}(t) : t \in [0, T]\}$ be a family of centered stationary Gaussian processes with

$$\mathbb{C}\mathrm{ov}(Y_{\varepsilon,u}(s), Y_{\varepsilon,u}(t)) = \mathrm{e}^{-(1-\varepsilon)Au^{-2}|s-t|^{\alpha+2t_u^\beta}},$$

for $\varepsilon \in (0, 1)$, $u > 0$ such that $\alpha + 2t_u^\beta \leq 2$ and $s, t, \in [0, T]$.
- $\{Z_{\varepsilon,u}(t) : t \in [0, T]\}$ be a family of centered stationary Gaussian processes with

$$\mathbb{C}\mathrm{ov}(Z_{\varepsilon,u}(s), Z_{\varepsilon,u}(t)) = \mathrm{e}^{-(1+\varepsilon)Au^{-2}|s-t|^{\alpha}},$$

for $\varepsilon \geq 0$, $u > 0$ and $s, t, \in [0, T]$.

Due to A1 (combined with Definition 1.1) α is strictly smaller than 2. This guarantees that covariance functions of $Y_{\varepsilon,u}(t)$ and $Z_{\varepsilon,u}(t)$ are positive-definite. Hence the introduced families of Gaussian processes exist.

The definition of $Y_{\varepsilon,u}(t)$ and $Z_{\varepsilon,u}(t)$ implies that uniformly with respect to $s, t \in [0, T]$

$$1 - \mathbb{C}\mathrm{ov}(Y_{\varepsilon,u}(s), Y_{\varepsilon,u}(t)) = (1-\varepsilon)Au^{-2}\,|s-t|^{\alpha+2t_u^\beta}\,(1+o(1)) \tag{5}$$

and

$$1 - \mathbb{C}\mathrm{ov}(Z_{\varepsilon,u}(s), Z_{\varepsilon,u}(t)) = (1+\varepsilon)Au^{-2}\,|s-t|^{\alpha}\,(1+o(1)) \tag{6}$$

as $u \to \infty$.

Lemma 4.1. *For each $\varepsilon \in (0, 1)$ there exists u_ε such that for each $u \geq u_\varepsilon$*
(i) $\mathbb{P}\left(\sup_{t\in B_{j,k}} X(t) > u\right) \geq \mathbb{P}\left(\sup_{t\in[0,T]} Y_{\varepsilon,u}(t) > u\right)$;
(ii) $\mathbb{P}\left(\sup_{t\in B_{j,k}} X(t) > u\right) \leq \mathbb{P}\left(\sup_{t\in[0,T]} Z_{\varepsilon,u}(t) > u\right)$,
for each $(j, k) \in \mathcal{U}$.

Proof. The idea of the proof is based on an appropriate application of Slepian's inequality (see, e.g. Theorem C.1 in [14]). Let $X_{j,k,u}(t) = X\left(a_k + \frac{jT+t}{u^{2/\alpha(a_{k+1})}}\right)$. Since $\sup_{t\in B_{j,k}} X(t) =_{\mathrm{d}} \sup_{t\in[0,T]} X_{j,k,u}(t)$, then it suffices to analyze $X_{j,k,u}(t)$.

Ad (i)

For sufficiently large u and $s, t \in [0, T]$ we have

$$\begin{aligned}
&1 - \mathbb{C}\mathrm{ov}(X_{j,k,u}(s), X_{j,k,u}(t))\\
&\quad= 1 - \mathbb{C}\mathrm{ov}\left(X(a_k + u^{-2/\alpha(a_{k+1})}(jT+s)), X(a_k + u^{-2/\alpha(a_{k+1})}(jT+t))\right)\\
&\quad\geq (1-\varepsilon/2)^{1/3} A \left|u^{-2/\alpha(a_{k+1})}(s-t)\right|^{\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)}\\
&\quad= (1-\varepsilon/2)^{1/3} A u^{-2\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)/\alpha(a_{k+1})}\,|s-t|^{\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)}\\
&\quad= (1-\varepsilon/2)^{1/3} A \times I_1 \times I_2.
\end{aligned} \tag{7}$$

We deal with I_1 and I_2 separately.

Ad I_1 For sufficiently large u, uniformly with respect to k,

$$\begin{aligned}
I_1 &= u^{-2\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)/\alpha(a_{k+1})} = u^{-2}u^{2\left(\alpha(a_{k+1})-\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)\right)/\alpha(a_{k+1})}\\
&= u^{-2}\mathrm{e}^{2\log(u)\left(\alpha(a_{k+1})-\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)\right)/\alpha(a_{k+1})}\\
&\geq u^{-2}(1-\varepsilon/2)^{1/3}.
\end{aligned} \tag{8}$$

Inequality (8) follows from the fact that, due to A2,

$$\begin{aligned}
&\log(u)\left|\alpha(a_{k+1})-\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)\right| \\
&\quad\le \log(u)\left(\left|(a_{k+1})^{\beta}-\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)^{\beta}\right|+2t_u^{\beta+\delta}\right) \\
&\quad\le \log(u)\left(\frac{1}{\log(u)(\log\log(u))^{1/\beta}}+2t_u^{\beta+\delta}\right) \\
&\quad\le \frac{1}{(\log\log(u))^{1/\beta}}+2\log(u)\left(\frac{\alpha^2}{\beta}\frac{\log\log(u)}{\log(u)}\right)^{(\beta+\delta)/\beta}\to 0,\quad \text{as } u\to\infty.
\end{aligned}$$

Ad I_2 We prove that

$$I_2 \ge (1-\varepsilon/2)^{1/3}\,|s-t|^{\alpha+2t_u^{\beta}}. \tag{9}$$

Assumption A2 implies that

$$\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right) < \alpha+2t_u^{\beta} \tag{10}$$

for each $(j,k)\in\mathcal{U}$. Thus if $|s-t|<1$, then (9) holds immediately.

If $1\le|s-t|\le T$, then due to (10)

$$\begin{aligned}
I_2 &= |s-t|^{\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)} \\
&\ge T^{\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)-\alpha-2t_u^{\beta}}\,|s-t|^{\alpha+2t_u^{\beta}} \\
&\ge T^{-2t_u^{\beta}}\,|s-t|^{\alpha+2t_u^{\beta}} \ge (1-\varepsilon/2)^{1/3}\,|s-t|^{\alpha+2t_u^{\beta}}
\end{aligned}$$

for sufficiently large u.

The combination of (7) with (8) and (9) gives that for sufficiently large u, uniformly with respect to $(j,k)\in\mathcal{U}$,

$$\begin{aligned}
1-\mathbb{C}\mathrm{ov}(X_{j,k,u}(s),X_{j,k,u}(t)) &\ge (1-\varepsilon/2)Au^{-2}|t-s|^{\alpha+2t_u^{\beta}} \\
&\ge 1-\mathbb{C}\mathrm{ov}(Y_{\varepsilon,u}(s),Y_{\varepsilon,u}(t)).
\end{aligned}$$

Thus the application of Slepian's inequality completes the proof of (i).

Ad (ii)

For sufficiently large u

$$\begin{aligned}
&1-\mathbb{C}\mathrm{ov}(X_{j,k,u}(s),X_{j,k,u}(t)) \\
&\quad= 1-\mathbb{C}\mathrm{ov}(X(a_k+u^{-2/\alpha(a_{k+1})}(jT+s)),X(a_k+u^{-2/\alpha(a_{k+1})}(jT+t))) \\
&\quad\le (1+\varepsilon)^{1/3}A\left|u^{-2/\alpha(a_{k+1})}(s-t)\right|^{\alpha\left(a_k+u^{-2/\alpha(a_{k+1})}(jT+t)\right)}.
\end{aligned}$$

Following the argument analogous to that for the proof of (i), we obtain that for sufficiently large u, uniformly with respect to k, and $s,t\in[0,T]$

$$1-\mathbb{C}\mathrm{ov}(X_{j,k,u}(s),X_{j,k,u}(t)) \le 1-\mathbb{C}\mathrm{ov}(Z_{\varepsilon,u}(s),Z_{\varepsilon,u}(t)).$$

Again the application of Slepian's inequality completes the proof. □

Lemma 4.2. *For any $T > 1$ and $\varepsilon \in (0, 1)$, as $u \to \infty$,*

(i) $\mathbb{P}\left(\sup_{t\in[0,T]} Y_{\varepsilon,u}(t) > u\right) = \mathcal{H}_\alpha\left[T(A(1-\varepsilon))^{1/\alpha}\right]\Psi(u)(1+o(1))$;
(ii) $\mathbb{P}\left(\sup_{t\in[0,T]} Z_{\varepsilon,u}(t) > u\right) = \mathcal{H}_\alpha\left[T(A(1+\varepsilon))^{1/\alpha}\right]\Psi(u)(1+o(1))$.

Proof. Since proofs of (i) and (ii) are similar, we focus on the argument that shows (i).

Following the definition of $Y_{\varepsilon,u}(t)$, for each $s, t \in [0, T]$

$$\lim_{u\to\infty} u^2\left[1 - \mathbb{C}\mathrm{ov}\left(Y_{\varepsilon,u}\left(t(A(1-\varepsilon))^{-1/\alpha}\right), Y_{\varepsilon,u}\left(s(A(1-\varepsilon))^{-1/\alpha}\right)\right)\right]$$
$$= \lim_{u\to\infty} (A(1-\varepsilon))^{1-\left(\alpha+2t_u^\beta\right)/\alpha}|s-t|^{\alpha+2t_u^\beta} = |s-t|^\alpha.$$

Moreover, for all $s, t \in [0, T]$, sufficiently large u and some constant $C > 0$

$$u^2\left[1 - \mathbb{C}\mathrm{ov}\left(Y_{\varepsilon,u}\left(t(A(1-\varepsilon))^{-1/\alpha}\right), Y_{\varepsilon,u}\left(s(A(1-\varepsilon))^{-1/\alpha}\right)\right)\right]$$
$$\le (A(1-\varepsilon))^{1-\left(\alpha+2t_u^\beta\right)/\alpha}|s-t|^{\alpha+2t_u^\beta} \le CT^{2\alpha}|s-t|^\alpha. \tag{11}$$

Inequality (11) follows from the fact that

$$|s-t|^{\alpha+2t_u^\beta} \le |s-t|^\alpha, \quad \text{if } |s-t| < 1$$

and

$$|s-t|^{\alpha+2t_u^\beta} \le T^{2\alpha} \le T^{2\alpha}|s-t|^\alpha, \quad \text{if } 1 \le |s-t| \le T.$$

Hence, by Lemma 7 in [11], we conclude that

$$\mathbb{P}\left(\sup_{t\in[0,T]} Y_{\varepsilon,u}(t) > u\right) = \mathbb{P}\left(\sup_{t\in[0,T(A(1-\varepsilon))^{-1/\alpha}]} Y_{\varepsilon,u}\left(t(A(1-\varepsilon))^{1/\alpha}\right) > u\right)$$
$$= \mathcal{H}_\alpha\left[T(A(1-\varepsilon))^{1/\alpha}\right]\Psi(u)(1+o(1)), \tag{12}$$

as $u \to \infty$. This completes the proof. □

Lemma 4.3. *Let $\{Z^{(1)}_{\varepsilon,u}(t) : t \in [0,T]\}$, $\{Z^{(2)}_{\varepsilon,u}(t) : t \in [0,T]\}$ be independent copies of $\{Z_{\varepsilon,u}(t) : t \in [0,T]\}$. Then there exists a positive constant C such that, as $u \to \infty$, we get the statement*

$$\mathbb{P}\left(\sup_{(s,t)\in[0,T]^2} \frac{1}{\sqrt{2}}\left(Z^{(1)}_{\varepsilon,u}(s) + Z^{(2)}_{\varepsilon,u}(t)\right) > u\right) = \mathcal{H}^2_\alpha[CT]\,\Psi(u)\,(1+o(1)).$$

Proof. This straightforwardly follows from Lemma 6.1 in [14]. □

Lemma 4.4. *Let $(j_1, k_1) \prec (j_2, k_2)$. For sufficiently large u*

$$\sqrt{N^{j_2,k_2}_{j_1,k_1}} \ge \mathrm{e}^{-1}\frac{u^{2/\alpha(a_{k_1}+1)}}{u^{2/\alpha(a_{k_2}+1)}} \tag{13}$$

uniformly for all $(j_1, k_1), (j_2, k_2) \in \mathcal{L}$ such that $N^{j_2,k_2}_{j_1,k_1} > 0$.

Proof. Let $(j_1, k_1) \prec (j_2, k_2)$ and $N_{j_1,k_1}^{j_2,k_2} > 0$. Due to A1 and A2, for sufficiently large u, we have

$$\begin{aligned} \frac{u^{2/\alpha(a_{k_1+1})}}{u^{2/\alpha(a_{k_2+1})}} &= \mathrm{e}^{\frac{2}{\alpha(a_{k_1+1})\alpha(a_{k_2+1})}\log(u)\left(\alpha(a_{k_2+1})-\alpha(a_{k_1+1})\right)} \\ &\le \mathrm{e}^{\frac{2}{\alpha(a_{k_1+1})\alpha(a_{k_2+1})}\log(u)\left(a_{k_2+1}^{\beta}-a_{k_1+1}^{\beta}+2t_u^{\beta+\delta}\right)} \\ &= \mathrm{e}^{\frac{2}{\alpha(a_{k_1+1})\alpha(a_{k_2+1})}\log(u)\left(\frac{k_2-k_1}{\log(u)(\log\log(u))^{1/\beta}}+2t_u^{\beta+\delta}\right)} \\ &\le \mathrm{e}^{k_2-k_1}. \end{aligned} \tag{14}$$

In order to get the statement we analyze three separate cases:

- ◇ If $k_1 = k_2$, then the thesis is obvious (since, by assumption, $N_{j_1,k_1}^{j_2,k_1} \ge 1$).
- ◇ If $k_2 = k_1 + 1$, then due to (14) we have

$$\sqrt{N_{j_1,k_1}^{j_2,k_2}} \ge 1 \ge \mathrm{e}^{-1}\frac{u^{2/\alpha(a_{k_1+1})}}{u^{2/\alpha(a_{k_2+1})}}$$

for sufficiently large u.

- ◇ If $k_2 \ge k_1 + 2$, then $N_{j_1,k_1}^{j_2,k_2}$ can be bounded from below by the number of intervals $B_{\cdot,k_1+1}$ where $(\cdot, k_1 + 1) \in \mathcal{L}$. Thus, for sufficiently large u,

$$\sqrt{N_{j_1,k_1}^{j_2,k_2}} \ge \sqrt{\frac{a_{k_1+2} - a_{k_1+1}}{T} u^{2/\alpha(a_{k_1+2})}} \ge u^{\frac{1}{2\alpha}}.$$

Combining this with $u^{\frac{1}{2\alpha}} \ge \mathrm{e}^{m(u)} \ge \mathrm{e}^{k_2-k_1}$ and (14), we get the statement. □

The next lemma plays a key role in the lower bound part of the proof of Theorem 2.1. The idea of the proof is adapted from the proof of Lemma 6.3 in [14].

Lemma 4.5. *There exist positive constants C, C_1 such that:*

(i) *uniformly with respect to u for $(j_1, k_1), (j_2, k_2) \in \mathcal{L}$, $(j_1, k_1) \prec (j_2, k_2)$ and $N_{j_1,k_1}^{j_2,k_2} > 0$*

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in B_{j_1,k_1}} X(t) > u; \sup_{t\in B_{j_2,k_2}} X(t) > u\right) \\ &\quad \le CT^2 \exp\left(-C_1 \left(N_{j_1,k_1}^{j_2,k_2}\right)^{\alpha/2} T^{\alpha}\right) \Psi(u)(1 + o(1)), \end{aligned}$$

as $u \to \infty$;

(ii) *uniformly with respect to u for $(j, k), (j + 1, k) \in \mathcal{L}$*

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in B_{j,k}} X(t) > u; \sup_{t\in B_{j+1,k}} X(t + \sqrt{T}u^{-2/\alpha(a_{k+1})}) > u\right) \\ &\quad \le CT^2 \exp(-C_1 T^{\alpha/2}) \Psi(u)(1 + o(1)), \end{aligned}$$

as $u \to \infty$;

(iii) *uniformly with respect to u for* $(n(k), k), (0, k+1) \in \mathcal{L}$

$$\mathbb{P}\left(\sup_{t\in B_{n(k),k}} X(t) > u; \sup_{t\in B_{0,k+1}} X(t+\sqrt{T}u^{-2/\alpha(a_{k+2})}) > u\right)$$
$$\leq CT^2 \exp(-C_1 T^{\alpha/2})\Psi(u)(1+o(1)),$$

as $u \to \infty$.

Proof. Since the idea of the proof is analogous to that for the proof of Lemma 6.3 in [14], we present only the main steps of the argument of part (i). The dependence of T is straightforwardly related with the distance between the analyzed intervals. Thus in (ii), (iii) we have $T^{\alpha/2}$ in the exponent.

Let

$$Y_u(s,t) = X_u^{(1)}(t) + X_u^{(2)}(s),$$

where $X_u^{(1)}(t) = X(a_{k_1} + (j_1 T + t)/u^{2/\alpha(a_{k_1+1})})$ and $X_u^{(2)}(s) = X(a_{k_2} + (j_2 T + s)/u^{2/\alpha(a_{k_2+1})})$. Then

$$\mathbb{P}\left(\sup_{t\in B_{j_1,k_1}} X(t) > u; \sup_{t\in B_{j_2,k_2}} X(t) > u\right) \leq \mathbb{P}\left(\sup_{(s,t)\in[0,T]^2} Y_u(s,t) > 2u\right). \tag{15}$$

Note that for sufficiently large u, uniformly for $(j_1,k_1), (j_2,k_2) \in \mathcal{L}$

$$\mathbb{V}\mathrm{ar}(Y_u(s,t)) = 4 - 2(1 - \mathbb{C}\mathrm{ov}(X(a_{k_1} + (j_1T+t)/u^{2/\alpha(a_{k_1+1})}),$$
$$X(a_{k_2} + (j_2T+s)/u^{2/\alpha(a_{k_2+1})}))) \geq 2.$$

Since for $s \in B_{j_1,k_1}$ and $t \in B_{j_2,k_2}$ we have $|s-t| \geq N_{j_1,k_1}^{j_2,k_2} \frac{T}{u^{2/\alpha(a_{k_2+1})}}$, there exists $C_2 > 0$ such that for sufficiently large u

$$\mathbb{V}\mathrm{ar}(Y_u(s,t)) \leq 4 - C_2 \left| N_{j_1,k_1}^{j_2,k_2} \frac{T}{u^{2/\alpha(a_{k_1+1})}} \right|^{\alpha(a_{k_2+1})}.$$

Thus, due to Lemma 4.4, for some $C_3 > 0$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(Y_u(s,t)) &\leq 4 - C_3 \left| \sqrt{N_{j_1,k_1}^{j_2,k_2}} \frac{T}{u^{2/\alpha(a_{k_2+1})}} \right|^{\alpha(a_{k_2+1})} \\ &= 4 - C_3 \left| \sqrt{N_{j_1,k_1}^{j_2,k_2}} T \right|^{\alpha(a_{k_2+1})} u^{-2} \leq 4 - C_3 \left| \sqrt{N_{j_1,k_1}^{j_2,k_2}} T \right|^{\alpha} u^{-2}\end{aligned}$$

uniformly with respect to u for $(j_1,k_1), (j_2,k_2) \in \mathcal{L}$.

Let $\overline{Y}_u(s,t) = Y_u(s,t)/\sqrt{\mathbb{V}\mathrm{ar}(Y_u(s,t))}$ and observe that

$$\mathbb{P}\left(\sup_{(s,t)\in[0,T]^2} Y_u(s,t) > 2u\right)$$
$$\leq \mathbb{P}\left(\sup_{(s,t)\in[0,T]^2} \overline{Y}_u(s,t) > \frac{2u}{\sqrt{4 - C_3 \left|\sqrt{N_{j_1,k_1}^{j_2,k_2}} T\right|^{\alpha} u^{-2}}}\right). \tag{16}$$

Moreover, following the argumentation analogous to that given in the proof of Lemma 6.3 in [14], for $s, s_1 \in B_{j_1,k_1}$ and $t, t_1 \in B_{j_2,k_2}$

$$\begin{aligned}
&\mathbb{E}(\overline{Y}_u(s,t) - \overline{Y}_u(s_1,t_1))^2 \\
&= \mathbb{E}\left(\frac{Y_u(s,t) - Y_u(s_1,t_1)}{\sqrt{\mathbb{V}\mathrm{ar}(Y_u(s,t))}} + Y_u(s_1,t_1)\left(\frac{1}{\sqrt{\mathbb{V}\mathrm{ar}(Y_u(s,t))}} - \frac{1}{\sqrt{\mathbb{V}\mathrm{ar}(Y_u(s_1,t_1))}}\right)\right)^2 \\
&\le \frac{2}{\mathbb{V}\mathrm{ar}(Y_u(s,t))}\left(\mathbb{E}\left(Y_u(s,t) - Y_u(s_1,t_1)\right)^2 + \left(\sqrt{\mathbb{V}\mathrm{ar}(Y_u(s,t))} - \sqrt{\mathbb{V}\mathrm{ar}(Y_u(s_1,t_1))}\right)^2\right) \\
&\le \mathbb{E}\left(Y_u(s,t) - Y_u(s_1,t_1)\right)^2 + \left(\sqrt{\mathbb{V}\mathrm{ar}(Y_u(s,t))} - \sqrt{\mathbb{V}\mathrm{ar}(Y_u(s_1,t_1))}\right)^2 \\
&\le 2\mathbb{E}\left(Y_u(s,t) - Y_u(s_1,t_1)\right)^2 \\
&\le 4\left(\mathbb{E}(X_u^{(1)}(s) - X_u^{(1)}(s_1))^2 + \mathbb{E}(X_u^{(2)}(t) - X_u^{(2)}(t_1))^2\right) \\
&\le \frac{1}{2}\left(\mathbb{E}\left(Z_{8,u}^{(1)}(s) - Z_{8,u}^{(1)}(s_1)\right)^2 + \mathbb{E}\left(Z_{8,u}^{(2)}(t) - Z_{8,u}^{(2)}(t_1)\right)^2\right),
\end{aligned}$$

where $Z_{8,u}^{(1)}(t), Z_{8,u}^{(2)}(t)$ are mutually independent copies of $Z_{\varepsilon,u}(t)$ with $\varepsilon = 8$.

Thus from Slepian's inequality and Lemma 4.3

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in[0,T]^2} \overline{Y}_u(s,t) > \frac{2u}{\sqrt{4 - C_3\left|\sqrt{N_{j_1,k_1}^{j_2,k_2}}T\right|^\alpha u^{-2}}}\right) \\
&\le \mathbb{P}\left(\sup_{(s,t)\in[0,T]^2} \frac{1}{\sqrt{2}}\left(Z_{8,u}^{(1)}(s) + Z_{8,u}^{(2)}(t)\right) > \frac{2u}{\sqrt{4 - C_3\left|\sqrt{N_{j_1,k_1}^{j_2,k_2}}T\right|^\alpha u^{-2}}}\right) \\
&= \mathcal{H}_\alpha^2[C_4T]\,\Psi\left(\frac{2u}{\sqrt{4 - C_3\left(\sqrt{N_{j_1,k_1}^{j_2,k_2}}T\right)^\alpha u^{-2}}}\right)(1+o(1)) \qquad (17) \\
&= \mathcal{H}_\alpha^2[C_4T]\frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{u^2}{2}\left(1 + \frac{C_3\left(\sqrt{N_{j_1,k_1}^{j_2,k_2}}T\right)^\alpha}{4u^2 - C_3\left(\sqrt{N_{j_1,k_1}^{j_2,k_2}}T\right)^\alpha}\right)\right)(1+o(1)) \\
&= \mathcal{H}_\alpha^2[C_4T]\frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{u^2}{2}\right)\exp\left(-C_1\left(N_{j_1,k_1}^{j_2,k_2}\right)^{\alpha/2}T^\alpha\right)(1+o(1)) \\
&= CT^2\exp\left(-C_1\left(N_{j_1,k_1}^{j_2,k_2}\right)^{\alpha/2}T^\alpha\right)\Psi(u)(1+o(1)), \qquad (18)
\end{aligned}$$

uniformly with respect to u for $(j_1, k_1), (j_2, k_2) \in \mathcal{L}$. In (17) we use that $\left|\sqrt{N_{j_1,k_1}^{j_2,k_2}}T\right|^{\alpha} u^{-2} \to 0$ as $u \to \infty$ uniformly for $(j_1, k_1), (j_2, k_2) \in \mathcal{L}$ (since $N_{j_1,k_1}^{j_2,k_2} \le t_u u^{2/\alpha}/T$).

The combination of (15) with (16) and (18) completes the proof. □

Lemma 4.6. *Let $X(t)$ be an $\alpha(t)$-locally stationary Gaussian process. If* A1 *and* A2 *are satisfied, then there exists a constant $C > 0$ such that for sufficiently large u*

$$\mathbb{P}\left(\sup_{t\in[t_u,S]} X(t) > u\right) \le C S u^{2/\alpha} \log^{-4/(3\beta)}(u)\, \Psi(u).$$

Proof. Let $b_u = u^{-2/\left(\alpha+\frac{3}{4}t_u^{\beta}\right)}$. Observe that there exists a constant $C_0 > 0$ such that for sufficiently large u and $|t-s| \le b_u$

$$1 - \mathbb{C}\mathrm{ov}(X(s), X(t)) < 1 - \mathrm{e}^{-C_0|s-t|^{\alpha+\frac{3}{4}t_u^{\beta}}}.$$

Let $\{\widetilde{X}_u(t) : t \ge 0\}$ be a family of centered stationary Gaussian processes such that

$$\mathbb{C}\mathrm{ov}(\widetilde{X}_u(t), \widetilde{X}_u(s)) = \mathrm{e}^{-C_0|s-t|^{\alpha+\frac{3}{4}t_u^{\beta}}}.$$

Then, from Slepian's inequality we get

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[t_u,S]} X(t) > u\right) &\le \sum_{i=0}^{\lfloor S b_u^{-1}\rfloor+1} \mathbb{P}\left(\sup_{t\in[ib_u,(i+1)b_u]} \widetilde{X}_u(t) > u\right)\\ &= (\lfloor S b_u^{-1}\rfloor + 1)\mathbb{P}\left(\sup_{t\in[0,b_u]} \widetilde{X}_u(t) > u\right),\end{aligned}$$

for sufficiently large u. Notice that for each $s, t \in [0, 1]$

$$1 - \mathbb{C}\mathrm{ov}(\widetilde{X}_u(b_u t), \widetilde{X}_u(b_u s)) = C_0 u^{-2}|s-t|^{\alpha+\frac{3}{4}t_u^{\beta}}(1+o(1)) = C_0 u^{-2}|s-t|^{\alpha}(1+o(1)).$$

Hence, from Lemma D.1 in [14],

$$\mathbb{P}\left(\sup_{t\in[0,b_u]} \widetilde{X}_u(t) > u\right) = \mathcal{H}_\alpha[1]\Psi(u)(1+o(1)),$$

as $u \to \infty$. Combining this with the fact that

$$\begin{aligned} b_u^{-1} &= u^{2/\left(\alpha+\frac{3}{4}t_u^{\beta}\right)} = u^{2/\alpha} u^{2/\left(\alpha+\frac{3}{4}t_u^{\beta}\right)-2/\alpha} = u^{2/\alpha} u^{-\frac{3}{2}t_u^{\beta}/\left(\alpha\cdot\left(\alpha+\frac{3}{4}t_u^{\beta}\right)\right)}\\ &= u^{2/\alpha} u^{-\frac{3}{2}\frac{\alpha^2 \log(\log(u))}{\beta\log(u)}/\left(\alpha\cdot\left(\alpha+\frac{3}{4}t_u^{\beta}\right)\right)} \le u^{2/\alpha} u^{-4/3\frac{\log(\log(u))}{\beta\log(u)}} = u^{2/\alpha}\log^{-4/(3\beta)}(u),\end{aligned}$$

the proof is completed. □

4.1. *Proof of Theorem 2.1*

As pointed out at the beginning of Section 4 we present only the proof for $t_0 = 0$.

Let $T > 1$, $0 < \varepsilon < 1$ be given. In view of Lemma 4.6 it suffices to focus on

$$\theta(u) = \mathbb{P}\left(\sup_{t\in[0,t_u]} X(t) > u\right).$$

1. (Upper bound)

Due to Lemma 4.1, for sufficiently large u, we get

$$\begin{aligned}\theta(u) &\le \sum_{(j,k)\in\mathcal{U}} \mathbb{P}\left(\sup_{t\in B_{j,k}} X(t) > u\right) \le \sum_{(j,k)\in\mathcal{U}} \mathbb{P}\left(\sup_{t\in[0,T]} Z_{\varepsilon,u} > u\right)\\ &= \sum_{k=0}^{m(u)}\sum_{j=0}^{n(k)} \mathbb{P}\left(\sup_{t\in[0,T]} Z_{\varepsilon,u} > u\right).\end{aligned}$$

Thus, using Lemma 4.2 and the definition of $n(k)$,

$$\begin{aligned}\theta(u) &\le \sum_{k=0}^{m(u)} \frac{a_{k+1}-a_k}{T} u^{2/\alpha(a_{k+1})}\mathcal{H}_\alpha\left[T(A(1+\varepsilon))^{1/\alpha}\right]\Psi(u)(1+o(1))\\ &= \frac{\mathcal{H}_\alpha\left[T(A(1+\varepsilon))^{1/\alpha}\right]}{T}\frac{u^{2/\alpha}}{\log^{1/\beta}(u)}\Psi(u)\\ &\quad\times\sum_{k=0}^{m(u)}\log^{1/\beta}(u)\,(a_{k+1}-a_k)\,\mathrm{e}^{\log(u)\left(\frac{2(\alpha-\alpha(a_{k+1}))}{\alpha\alpha(a_{k+1})}\right)}(1+o(1))\\ &\le \frac{\mathcal{H}_\alpha\left[T(A(1+\varepsilon))^{1/\alpha}\right]}{T}\frac{u^{2/\alpha}}{\log^{1/\beta}(u)}\Psi(u)\\ &\quad\times\sum_{k=0}^{m(u)}\log^{1/\beta}(u)\,(a_{k+1}-a_k)\,\mathrm{e}^{\frac{-2(1-\varepsilon)}{\alpha^2}\log(u)\left(a_{k+1}^{\beta}-a_{k+1}^{\beta+\delta}\right)}(1+o(1)) \qquad (19)\\ &\le \frac{\mathcal{H}_\alpha\left[T(A(1+\varepsilon))^{1/\alpha}\right]}{T}\frac{u^{2/\alpha}}{\log^{1/\beta}(u)}\Psi(u) \qquad (20)\\ &\quad\times\sum_{k=0}^{m(u)}\log^{1/\beta}(u)\,(a_{k+1}-a_k)\,\mathrm{e}^{\frac{-2(1-\varepsilon)}{\alpha^2}\left(\log^{1/\beta}(u)a_{k+1}\right)^{\beta}}\mathrm{e}^{\frac{2(1-\varepsilon)}{\alpha^2}\log(u)a_{m(u)+1}^{\beta+\delta}}(1+o(1))\end{aligned}$$

as $u\to\infty$, where inequality (19) follows from A2. Recall that due to Theorem D.2 in [14]

$$\lim_{T\to\infty}\frac{\mathcal{H}_\alpha\left[T(A(1+\varepsilon))^{1/\alpha}\right]}{T} = (A(1+\varepsilon))^{1/\alpha}\mathcal{H}_\alpha. \qquad (21)$$

Moreover, using that $a_{m(u)} \le t_u$ and $\lim_{u\to\infty}\log(u)t_u^{\beta+\delta} = 0$ we have

$$\lim_{u\to\infty}\mathrm{e}^{\frac{2(1-\varepsilon)}{\alpha^2}\log(u)a_{m(u)+1}^{\beta+\delta}} = 1. \qquad (22)$$

Finally, after checking that $\log^{1/\beta}(u)\,(a_{k+1}-a_k) \to 0$ uniformly with respect to $k = 0,\ldots,m(u)$ and $\log^{1/\beta}(u)a_{m(u)+1} \to \infty$, as $u \to \infty$, we have

$$\lim_{u\to\infty}\sum_{k=0}^{m(u)}\log^{1/\beta}(u)\,(a_{k+1}-a_k)\,\mathrm{e}^{\frac{-2(1-\varepsilon)}{\alpha^2}\left(\log^{1/\beta}(u)a_{k+1}\right)^{\beta}} = \int_0^{\infty}\mathrm{e}^{-(1-\varepsilon)2x^{\beta}/\alpha^2}\mathrm{d}x$$

$$= \left(\frac{\alpha^2}{2(1-\varepsilon)}\right)^{\frac{1}{\beta}}\frac{\Gamma(1/\beta)}{\beta}. \tag{23}$$

Applying (21), (22) and (23) to (20) and letting $T \to \infty$ we get that the term in Eq. (20) can be asymptotically (for large u) bounded by

$$\mathcal{H}_\alpha\,(A(1+\varepsilon))^{1/\alpha}\left(\frac{\alpha^2}{2(1-\varepsilon)}\right)^{1/\beta}\frac{\Gamma(1/\beta)}{\beta}\frac{u^{2/\alpha}}{\log^{1/\beta}(u)}\Psi(u)(1+o(1)).$$

Thus letting $\varepsilon \to 0$ we obtain the desired asymptotic upper bound.

2. (Lower bound)

Using Bonferroni's inequality we get

$$\theta(u) \geq \sum_{(j,k)\in\mathcal{L}}\mathbb{P}\left(\sup_{t\in B_{j,k}}X(t) > u\right)$$

$$-2\sum_{\substack{(j_1,k_1),(j_2,k_2)\in\mathcal{L}\\(j_1,k_1)\prec(j_2,k_2)}}\mathbb{P}\left(\sup_{t\in B_{j_1,k_1}}X(t) > u;\ \sup_{t\in B_{j_2,k_2}}X(t) > u\right). \tag{24}$$

From Lemma 4.1, we get that for sufficiently large u

$$\sum_{(j,k)\in\mathcal{L}}\mathbb{P}\left(\sup_{t\in B_{j,k}}X(t) > u\right) \geq \sum_{k=0}^{m(u)-1}\sum_{j=0}^{n(k)-1}\mathbb{P}\left(\sup_{t\in[0,T]}Y_{\varepsilon,u} > u\right).$$

Thus the combination of (i) of Lemma 4.2 with the same chain of argumentation as in the proof of the upper bound gives the lower asymptotics, which is tight with the asymptotics obtained in the preceding part of the proof.

Hence it is enough to prove that

$$\lim_{u\to\infty}\frac{\sum_{\substack{(j_1,k_1),(j_2,k_2)\in\mathcal{L}\\(j_1,k_1)\prec(j_2,k_2)}}\mathbb{P}\left(\sup_{t\in B_{j_1,k_1}}X(t) > u;\ \sup_{t\in B_{j_2,k_2}}X(t) > u\right)}{\frac{u^{2/\alpha}}{\log^{1/\beta}(u)}\Psi(u)} = 0,$$

which, in view of Lemma 4.5, follows line by line the same argumentation as that for the estimation of the double sum in the proof of Theorem D.2 in Piterbarg [14]. This completes the proof. □

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